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Optimal Design of Batch-Storage Network Under Joint Uncertainties

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This study sought to find analytic solutions to the problem of determining the optimal capacity of a batch-storage network to meet demand for finished products in a system undergoing joint random variations of operating time and batch quantity. The raw material purchasing flow and final product demand flow are susceptible to joint random variations in the order cycle time and batch size. The production processes also have joint random variations in production cycle time and product quantity. Waste regeneration or disposal processes are included into the network to treat the spoiled materials from failed batches. The objective function of the optimization is minimizing the expected total cost, which is composed of setup and inventory holding costs as well as the capital costs of constructing processes and storage units. A production and inventory method, the PSW (periodic square wave) model, provides a unique graphical method to find the upper/lower bounds and average of random flows, which are used to construct terms of the objective function and constraints of the optimization model. The advantage of this model is that it provides a set of simple analytic solutions while also maintaining a realistic description of the random material flows between processes and storage units; as a consequence of these analytic solutions, the computation burden is significantly reduced. The proposed method has the potential to rapidly provide very useful data on base investment decisions during the early plant design stage. It should be particularly useful when these decisions must be made in a highly uncertain business environment. *AIChE J,* 54: 2567–2580, 2008
Keywords: optimal lot size, batch-storage network, joint uncertainty, waste treatment process, analytical solution

Introduction

The production and inventory analysis method known as the periodic square wave (PSW) method was recently developed and used to determine the optimal design of a parallel batch-storage system.[1] Subsequently, the method was extended to handle a sequential multistage batch-storage network (BSN),[2] and further modified to handle a nonsequential network structure that can deal with recycle material flows in the network.[3] The key advantage of the PSW model over other models lies in its simple analytical sizing and timing equations. This advantageous characteristic has been exploited in an analysis of an integrated financial and production system.[4,5] The range of physical process structures for which the PSW model can be used has expanded from batch processes to multitasking batch or semicontinuous processes[6] and a multisite batch production/transportation network (i.e., a large-scale supply chain system).[7] A BSN basically resembles a state task network (STN) in which batch corresponds to task and storage corresponds to state; thus, BSNs provide an effective representation of supply chains such as purchasing, production, transportation, and demand processes.

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One of the challenging problems in supply chain optimization is dealing with uncertainty. Major business uncertainties arise from product demand forecasting, production equipment malfunctions, off-spec materials, changes in the prices of raw materials or finished products, and raw material supply shortages. A promising approach to account for such uncertainties is to include a computational mechanism to mitigate such uncertainty effects into the supply chain optimization model. A review paper for optimization under uncertainty has been published.[8] Recent research in this area can be divided into various categories: simulation-based optimization,[9] scenario-based optimization,[7] chance-constrained programming,[10] multistage stochastic programming with recourse,[11] approximate dynamic programming,[12] multiparametric programming,[13] genetic algorithm with mathematical programming,[14] robust optimization with bounded uncertainty,[15] a fuzzy optimization model,[16] and a branch and cut algorithm that uses Lagrangean decomposition.[17] However, the models developed to date are unsuitable for use in real applications due to severe computational complexity arising from the treatment of uncertainty as well as unknown probabilistic parameters. Given this situation, the present study aims to develop a compact analytical solution to optimize the design and/or operation of large-scale supply chains with batch processes. In this study, a novel optimization model resulting in simple analytical solutions with negligible computational burden is introduced.

A previous study using the PSW model on a BSN developed analytical solutions of supply chain optimization dealt with uncertainty.[18] The sources of uncertainty were batch size and cycle time variations in raw material purchasing, batch production, transportation, and finished product demand processes. The processes were classified into three types according to their random characteristics: (1) processes that possess uncertainty only in the cycle time; (2) processes that possess uncertainty only in the batch size; and (3) processes that possess joint uncertainty in both the cycle time and batch size. In modern society, batch material losses associated with raw material purchasing and transportation processes occur infrequently; therefore, these processes were exclusively considered as Type 1 processes in the previous study. Production processes may be either Type 1 or Type 2 processes. Mixing or blending processes do not usually involve batch material loss, and thus were considered to Type 1 processes. Batch material losses do, however, occur in many reaction processes; hence these processes were considered Type 2 processes. Most processes had joint uncertainties in cycle time and batch size (Type 3 process); however, such joint uncertainties were excluded in the previous study due to the complexity of handling them. In the present study, an approach is developed that overcomes the complexity of handling joint uncertainties of cycle time and batch size. Here it is considered that all processes are subject to joint uncertainties of cycle time and batch size; therefore all processes are Type 3 processes. Under this scheme, Type 1 and 2 processes are subsystems of Type 3 and hence do not need to be considered separately. In spite of the increased problem complexity associated with this approach, analytical solutions for the optimization problem are still available. These analytical solutions greatly reduce the computation time.

When a batch production process is susceptible to random failures, the volume of on-spec product material fluctuates randomly for a given feed volume. In other words, a random amount of waste material is produced, as the amount of waste material equals feed volume minus on-spec product volume. Therefore, the system that deals with material quantity uncertainty should include waste material storage units and waste disposal processes. In this study, the BSN is modified to include a waste material stream in a batch process connected with a waste material storage unit and a waste disposal process connected with a waste material storage.

A multiperiod formulation to treat long-term trends of variables and parameters that was introduced in the previous study[18] is omitted in this study and, therefore, the derivation corresponds to single-period formulation that accounts for short-term variations of variables and parameters. The single-period formulation presented here can be easily expanded to the multiperiod formulation, as was done in the previous study.[18]

The remainder of this article is organized as follows. First, the upper/lower bounds and average of the batch flow between a storage unit and a process exposed to joint uncertainties are found using basic probability theory and a unique graphical method. The upper/lower bounds and average of the batch flow are used as constraints or terms of the objective function in the subsequent optimization model. An optimization model that minimizes the expected sum of various costs under the constraints of no depletion of inventory and material balance is next introduced and the analytical solutions of the Kuhn-Tucker conditions of the optimization model are found. Finally, some computational results and sensitivity analysis with respect to some input parameters that highlight the advantages of the proposed approach are given, together with the conclusions of the work.

Upper and Lower Bounds of a Flow with Joint Uncertainties

Figure 1 shows three types of uncertainties: (a) uncertainty only in cycle time, (b) uncertainty only in batch size, and (c) joint uncertainties in both cycle time and batch size. The focus of the present study is joint uncertainties. The random properties of joint uncertainties are characterized by two random variables, $\mathbf{B}_{(\mathbf{l})}$ and $\boldsymbol{\omega}_{(\mathbf{l})}$, as shown in Figure 1c, where subscript **(l)** represents the sequence of batch occurrence. It is not necessary to know the exact distribution functions of $\mathbf{B}_{(\mathbf{l})}$ and $\boldsymbol{\omega}_{(\mathbf{l})}$. It is assumed that $\mathbf{B}_{(\mathbf{l})}$ has a symmetrical distribution function with $\underline{B} \leq \mathbf{B}_{(\mathbf{l})} \leq \overline{\overline{B}}$ and $\boldsymbol{\omega}_{(\mathbf{l})}$ has a nonsymmetrical distribution function with $\underline{\omega} \leq \boldsymbol{\omega}_{(\mathbf{l})}$. The mean values of $\mathbf{B}_{(\mathbf{l})}$ and $\boldsymbol{\omega}_{(\mathbf{l})}$ are $\overline{B} \equiv (\overline{\overline{B}} + \underline{B})/2$ and $\overline{\omega}$, respectively. Two design parameters, time availability α and size availability β, are introduced such that $\alpha \equiv \underline{\omega}/\overline{\omega}$ and $\beta \equiv \underline{B}/\overline{B} \equiv 2 - (\overline{\overline{B}}/\overline{B})$, where $0 < \alpha \leq 1$ and $0 \leq \beta \leq 1$. As α and/or β approach 1, the process becomes more deterministic. The concept of availability, defined as minimum value without failure divided by average value with failure, comes from failure modes and effects analysis (FMEA).[19] Note that the process with $\beta = 1$ and $0 < \alpha < 1$ is Type 1, and the process with $\alpha = 1$ and $0 < \beta < 1$ is Type 2 in the previous study.[18] Suppose that $\mathbf{B}_{(\mathbf{l})}$ and $\boldsymbol{\omega}_{(\mathbf{l})}$ have identical independent distribution functions with respect to **(l)**. For given conver-

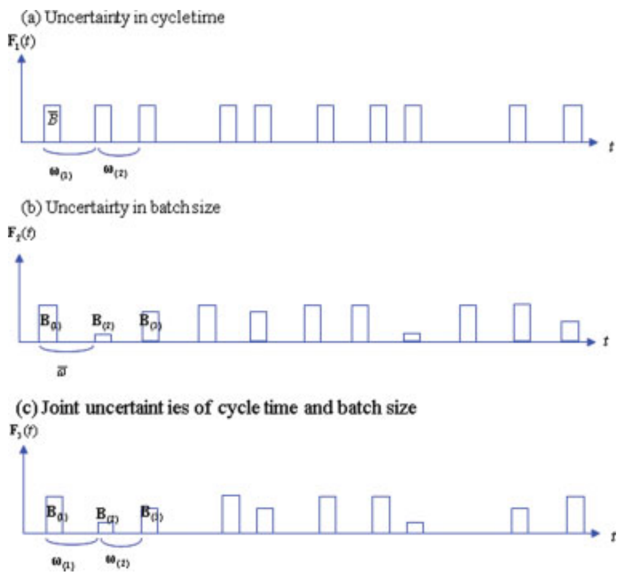

Figure 1. Types of uncertainty.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

gence limits $0 < \varepsilon_1,\varepsilon_2 \ll 1$ and confidence levels $0 < \delta_1$, $\delta_2 \ll 1$, the weak law of large numbers says that there exists an integer η such that $P\{|\frac{1}{\eta}\sum_{l=1}^{\eta}\mathbf{B}_{(l)} - \overline{B}| < \varepsilon_1\} \geq 1 - \delta_1$ and $P\{|\frac{1}{\eta}\sum_{l=1}^{\eta}\boldsymbol{\omega}_{(l)} - \overline{\omega}| < \varepsilon_2\} \geq 1 - \delta_2$. From Tchebycheff's inequality, $\eta \geq \mathrm{Var}(\mathbf{B}_{(l)})/\delta_1{\varepsilon_1}^2$ and $\eta \geq \mathrm{Var}(\boldsymbol{\omega}_{(l)})/\delta_2{\varepsilon_2}^2$, that is, $\eta = \max\{\mathrm{int}[\mathrm{Var}(\mathbf{B}_{(l)})/\delta_1{\varepsilon_1}^2],\ \mathrm{int}[\mathrm{Var}(\boldsymbol{\omega}_{(l)})/\delta_2{\varepsilon_2}^2]\} + 1$ if the least integer is chosen.[18] Where Var(.) is variance operator and int[.] is a truncation function to make integer. The parameter η, called the occurrence number, should be an even number in order for 0.5η to be an integer. The time interval during which η batches occur is defined as the long cycle time $\tilde{\omega}$. Because the sample means of $\mathbf{B}_{(l)}$ and $\boldsymbol{\omega}_{(l)}$ converge to their mean values during the long cycle time according to the weak law of large numbers, $\tilde{\omega} = \eta\overline{\omega} = \frac{\eta}{\alpha}\underline{\underline{\omega}}$. Here, the long cycle time corresponds to the least period within which all random effects diminish with a given confidence level. Two more parameters are introduced for convenience, the average flow rate $D \equiv \overline{B}/\overline{\omega} = \alpha\overline{B}/\underline{\underline{\omega}}$ and the total dead time within a long cycle time $d \equiv \tilde{\omega} - \eta\underline{\underline{\omega}} = (\frac{1}{\alpha} - 1)\eta\underline{\underline{\omega}}$.

To generate the optimization formulation, the upper/lower bounds and average inventory level of storage units under joint uncertainties are needed. The upper bound of the inventory level will be used to compute the storage size; the lower bound of the inventory level will be used in the optimization constraint that ensures that the inventory level is always nonnegative; and the average of the inventory level will be used to compute the inventory holding cost of the optimization problem. If the upper/lower bounds and average of all flows coming into and going out of the storage units are known, the upper/lower bounds and average of the inventory level of the storage units can be easily identified. Note that the flow has a constant average flow rate D measured during a long cycle time. This means that in spite of the randomness, the total quantity processed during a long cycle time is constant. Two extreme cases of the flow with joint uncertainties

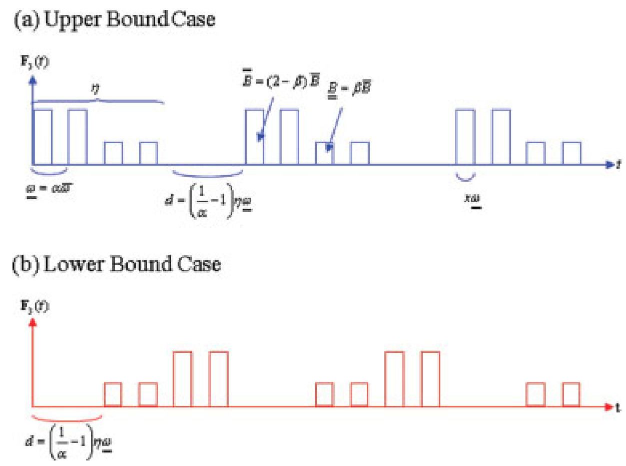

Figure 2. Two extreme cases of flow.

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exist—(a) the Upper bound case and (b) the Lower bound case-as shown in Figure 2 for the case of $\eta = 4$. The upper bound case has 0.5η times of maximum batch size $\overline{\overline{B}}$ with minimum cycle time $\underline{\underline{\omega}}$, 0.5η times of minimum batch size $\underline{\underline{B}}$ with minimum cycle time $\underline{\underline{\omega}}$, and a total dead time d within repeated long cycle times. The lower bound case has total dead time d, 0.5η times of minimum batch size $\underline{\underline{B}}$ with minimum cycle time $\underline{\underline{\omega}}$, and 0.5η times of maximum batch size $\overline{\overline{B}}$ with minimum cycle time $\underline{\underline{\omega}}$ within repeated long cycle times. Note that in spite of the greater difference between these two cases, the total quantity processed during a long cycle time of both cases is a constant, $D\tilde{\omega}$. Figure 3 shows the cumulative flow functions of the two cases. The dotted lines are the upper and lower bounds of the two extreme cases. Note that there are two contacting points depending on the values of α and β. The integral of all flows with joint uncertainties exists between the dotted lines, that is, $\underline{\underline{\mathrm{UPSW}}} \leq \int_0 \mathbf{F}_3(t)dt \leq \overline{\overline{\mathrm{UPSW}}}$, where

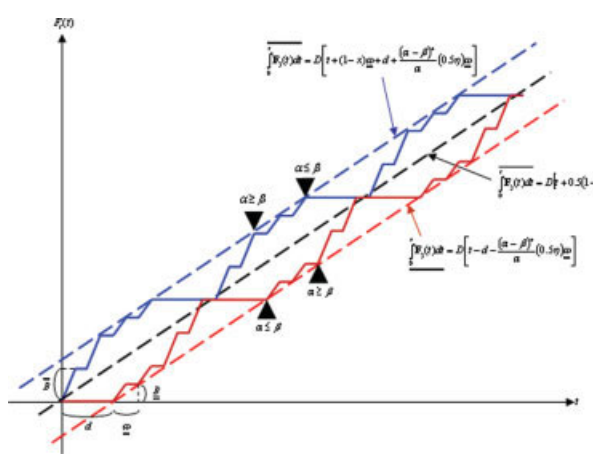

Figure 3. Cumulative flow functions for two extreme cases.

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$$\underline{\mathrm{UPSW}}(t;D,\underline{\underline{\omega}},t',x,\theta) = D[t-t'-\theta\underline{\underline{\omega}}] \quad (1)$$

$$\overline{\overline{\mathrm{UPSW}}}(t;D,\underline{\underline{\omega}},t',x,\theta) = D[t-t'+(1-x)\underline{\underline{\omega}}+\theta\underline{\underline{\omega}}] \quad (2)$$

Here

$$\theta = \left(\frac{1}{\alpha}-1\right)\eta + \frac{(\alpha-\beta)^{+}}{\alpha}(0.5\eta) \quad (3)$$

where $(X)^{+} \equiv \max\{0,X\}$, x is called storage operation time fraction and t' is initial start-up time.

The average inventory level is highly dependent on the random properties of failures. The exact value of the average inventory level cannot be obtained without defining the probability distribution functions of all random variables, which is a nontrivial task. In this study, an intuitive approach is taken. Specifically, the average of flow $\overline{\mathrm{UPSW}} \equiv \overline{\int_0^t \mathbf{F}_3(\mathbf{t})dt}$ is selected as the line equidistant from the upper and lower bounds[18]:

$$\overline{\mathrm{UPSW}}(t;D,\underline{\underline{\omega}},t',x) = D[t-t'+0.5(1-x)\underline{\underline{\omega}}] \quad (4)$$

This selection could be the most probable.[18]

Optimization Model

A chemical plant that converts raw materials into final products through multiple physicochemical processing steps is composed of a set of storage units (J) and a set of batch processes (I), as shown in Figure 4. Note that storage index $j \in J$ is written as a superscript whereas process index $i \in I$ is written as a subscript. Transportation processes are considered a subset of batch processes without loss of generality.[7] Each storage unit is involved in six types of material movement: purchasing from suppliers ($k \in K(j)$), shipping in response to consumer demand ($m \in M(j)$), discharging to waste disposal sinks ($n \in N(j)$), feeding to production processes, producing from production processes, and producing waste materials from production processes.

Figure 5 shows a typical configuration of a process and Figure 6 shows the flows of that process. The feed flow to this process is assumed to have constant (maximum) batch size $\overline{\overline{B}}_i$ and uncertain cycle time $\boldsymbol{\omega}_{\mathbf{i(l)}}$. The time availability of the feed flow to the process is α_i and the size availability of the feed flow to the process is 1, which means that there is no uncertainty in batch size. However, the batch size of the flows discharging from the process is random. (For simplicity, it is assumed that the processing does not change the material density.) The successful discharging batch quantity $\mathbf{B}'_{\mathbf{i(l)}}(\overline{\overline{B}}_i \geq \mathbf{B}'_{\mathbf{i(l)}} \geq \underline{\underline{B}}_i)$ goes to a product storage unit and the

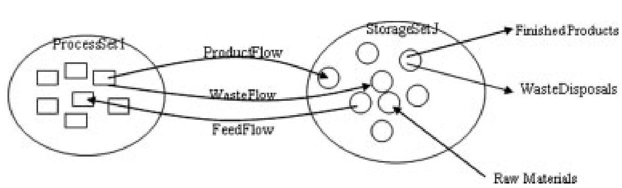

Figure 4. General structure of batch-storage network—process and storage sets.

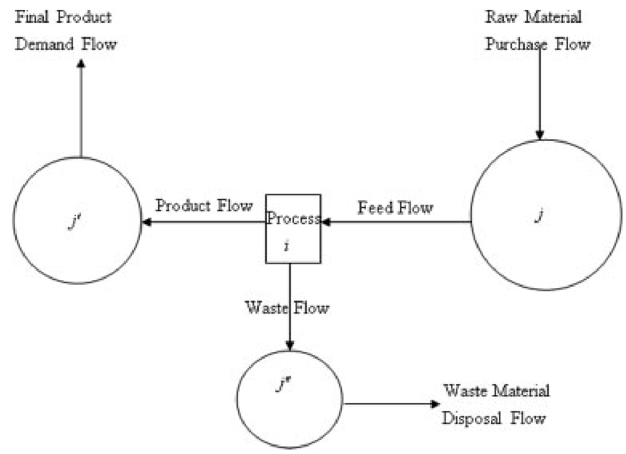

Figure 5. Process configuration with waste flow.

failed discharging batch quantity $\overline{\overline{B}}_i - \mathbf{B}'_{\mathbf{i(l)}}$ goes to a waste storage unit, as shown in Figures 5 and 6. The time availability of both product and waste flows discharging from the process is α_i, which is the same as that of the feed flow to the process. The size availability of the product flow discharging from the process is given by β_i. The size availability of the waste flow discharging from the process is 0.

Each production process requires multiple feedstock materials of fixed composition (f_i^j) and produces multiple products with fixed product yield (g_i^j). The spoiled material from a failed batch goes to storage units distinct from those where the product goes, according to the fixed waste yield ($\hat{g}_i^j$). Note that, for the same storage unit j and process i, $g_i^j \hat{g}_i^j = 0$.

In the absence of failure or uncertainty, the material flow from process to storage (or from storage to process) is represented by the deterministic PSW model. Each production

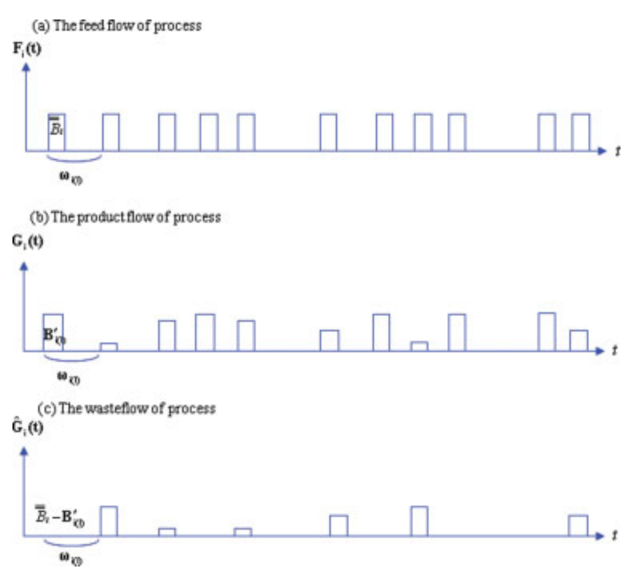

Figure 6. Flows of production process.

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process is supposed to produce a batch of product during every cycle time ω_i. The cycle time of a production unit is composed of the feedstock feeding time ($\grave{x}_i\omega_i$), processing time ($[1 - x'_i - \grave{x}_i]\omega_i$), and product discharging time ($x'_i\omega_i$), where $0 \le x'_i, \grave{x}_i \le 1$ are called storage operation time fractions. Note that the back prime mark on the variable indicates that the variable is defined for the feeding flow to a production process and the prime mark on the variable indicates that the variable is defined for the discharging flow from a production process. The processing is initiated at the start-up time $\grave{t}_i$ (or t'_i). Therefore, the deterministic material flow representation of the PSW model for a production process is composed of four variables: the batch size B_i, the cycle time ω_i, the storage operation time fraction $\grave{x}_i$ (or x'_i), and the start-up time $\grave{t}_i$ (or t'_i). The deterministic material flows of raw material purchased, waste disposal, and finished product demand are also each represented by four variables: $B^j_k, \omega^j_k, x^j_k, t^j_k$, $B^j_n, \omega^j_n, x^j_n, t^j_n$, and $B^j_m, \omega^j_m, x^j_m, t^j_m$, respectively. Among these variables, $\omega_i, \omega^j_k, \omega^j_n$, ω^j_m and B_i, B^j_k, B^j_n, B^j_m will be considered as random variables. For convenience, the variables without superscripts and subscripts, B, ω, and x, will be used to represent the batch size, cycle time, and storage operation time fraction of any process in raw material purchase, production, waste disposal, or finished product demand.

The following timing relationship exists between the start-up time of the feedstock streams and the start-up time of the product or waste streams of production processes:

$$t'_i = \grave{t}_i + \Delta t_i(.) \tag{5}$$

where $\Delta t_i(.)$ is a function of arbitrary variables. $D_i \equiv \overline{B}_i/\overline{\omega}_i = \alpha_i \overline{B}_i/\underline{\underline{\omega_i}}$ is the average feed flow rate through processes i. $\frac{D_i}{2-\beta_i} = \overline{B}_i/\overline{\omega}_i$ and $(1-\beta_i)/(2-\beta_i)D_i$ are the average product flow rate through process i and the average waste flow rate through process i respectively. The average material flow rates of raw material purchasing, waste disposal, and finished product demand are denoted by $D^j_k = \alpha^j_k \overline{B}^j_k/\underline{\underline{\omega^j_k}}$, $D^j_n = \alpha^j_n \overline{B}^j_n/\underline{\underline{\omega^j_n}}$, and $D^j_m = \alpha^j_m \overline{B}^j_m/\underline{\underline{\omega^j_m}}$, respectively. Hereinafter, the average flow rates will be used instead of batch sizes in most equations. The overall material balance around a storage unit results in the following relationships:

$$\sum_{i=1}^{|I|} \frac{1}{2-\beta_i} g^j_i D_i + \sum_{i=1}^{|I|} \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i + \sum_{k=1}^{|K(j)|} D^j_k = \sum_{i=1}^{|I|} f^j_i D_i + \sum_{m=1}^{|M(j)|} D^j_m + \sum_{n=1}^{|N(j)|} D^j_n \tag{6}$$

The upper bound of the inventory level, $\overline{\overline{V^j}}$, is computed by adding the upper bounds of all incoming flows and subtracting the lower bounds of all outgoing flows from the initial inventory. The lower bound of the inventory level, $\underline{\underline{V^j}}$, is computed by adding the lower bounds of all incoming flows and subtracting the upper bounds of all outgoing flows from the initial inventory. Incoming flows are raw material purchase, waste flows discharging from production processes, and product flows discharging from production processes. Outgoing flows are feed flows to production processes, waste disposal flows, and finished product demand flows. Using Eqs. 1 and 2, the upper and lower bounds of the inventory level can be expressed as follows:

$$\begin{aligned}\overline{\overline{V^j}} = V^j(0) &+ \sum_{k=1}^{|K(j)|} \overline{\overline{\mathrm{UPSW}}}(t; D^j_k, \omega^j_k, t^j_k, x^j_k, \theta^j_k) \\ &+ \sum_{k=1}^{|I|} \overline{\overline{\mathrm{UPSW}}}\left(t; \frac{1}{2-\beta_i} g^j_i D_i, \omega_i, t'_i, x'_i, \theta'_i\right) \\ &+ \sum_{i=1}^{|I|} \overline{\overline{\mathrm{UPSW}}}\left(t; \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i, \omega_i, t'_i, x'_i, \hat{\theta}_i\right) \\ &- \sum_{i=1}^{|I|} \underline{\underline{\mathrm{UPSW}}}(t; f^j_i D_i, \omega_i, \grave{t}_i, \grave{x}_l, \grave{\theta}_i) \\ &- \sum_{m=1}^{|M(j)|} \underline{\underline{\mathrm{UPSW}}}(t; D^j_m, \omega^j_m, t^j_m, x^j_m, \theta^j_m) \\ &- \sum_{n=1}^{|N(j)|} \underline{\underline{\mathrm{UPSW}}}(t; D^j_n, \omega^j_n, t^j_n, x^j_n, \theta^j_n)\end{aligned} \tag{7}$$

$$\begin{aligned}\underline{\underline{V^j}} = V^j(0) &+ \sum_{k=1}^{|K(j)|} \underline{\underline{\mathrm{UPSW}}}(t; D^j_k, \omega^j_k, t^j_k, x^j_k, \theta^j_k) \\ &+ \sum_{i=1}^{|I|} \underline{\underline{\mathrm{UPSW}}}\left(t; \frac{1}{2-\beta_i} g^j_i D_i, \omega_i, t'_i, x'_i, \theta'_i\right) \\ &+ \sum_{i=1}^{|I|} \underline{\underline{\mathrm{UPSW}}}\left(t; \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i, \omega_i, t'_i, x'_i, \hat{\theta}_i\right) \\ &- \sum_{i=1}^{|I|} \overline{\overline{\mathrm{UPSW}}}(t; f^j_i D_i, \omega_i, \grave{t}_i, \grave{x}_l, \grave{\theta}_i) \\ &- \sum_{m=1}^{|M(j)|} \overline{\overline{\mathrm{UPSW}}}(t; D^j_m, \omega^j_m, t^j_m, x^j_m, \theta^j_m) \\ &- \sum_{n=1}^{|N(j)|} \overline{\overline{\mathrm{UPSW}}}(t; D^j_n, \omega^j_n, t^j_n, x^j_n, \theta^j_n)\end{aligned} \tag{8}$$

Note that the feed flow to the process has a size availability of 1 and therefore $\grave{\theta}_i = (\frac{1}{\alpha_i} - 1)\eta_i$. The waste flow discharging from the process has a size availability of 0 and therefore $\hat{\theta}_i = (\frac{1}{\alpha_i} - 1)\eta_i + (0.5\eta_i)$. All of the other theta-values $\theta^j_k, \theta'_i, \theta^j_m$, and θ^j_n have the form of Eq. 3 with suitable superscripts and/or subscripts. For example, $\theta'_i = (\frac{1}{\alpha_i} - 1)\eta_i + (\alpha_i - \beta_i)^+/\alpha_i(0.5\eta_i)$. Then, Eqs. 7 and 8 are further developed by using Eqs. 1 and 2.

$$\begin{aligned}\overline{\overline{V^j}} = V^j(0) &+ \sum_{k=1}^{|K(j)|} (1 - x^j_k + \theta^j_k) D^j_k \underline{\underline{\omega^j_k}} - \sum_{k=1}^{|K(j)|} D^j_k t^j_k \\ &+ \sum_{i=1}^{|I|} (1 - x_i + \hat{\theta}_i) \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i \underline{\underline{\omega_i}} - \sum_{i=1}^{|I|} \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i [\grave{t}_i + \Delta t_i] \\ &+ \sum_{i=1}^{|I|} (1 - x'_i + \theta'_i) \frac{1}{2-\beta_i} g^j_i D_i \underline{\underline{\omega_i}} - \sum_{i=1}^{|I|} \frac{1}{2-\beta_i} g^j_i D_i [\grave{t}_i + \Delta t_i] \\ &+ \sum_{i=1}^{|I|} f^j_i D_i \grave{t}_i + \sum_{i=1}^{|I|} f^j_i D_i \grave{\theta}_i \underline{\underline{\omega_i}} + \sum_{m=1}^{|M(j)|} D^j_m t^j_m + \sum_{m=1}^{|M(j)|} D^j_m \theta^j_m \underline{\underline{\omega^j_m}} \\ &+ \sum_{n=1}^{|N(j)|} D^j_n t^j_n + \sum_{n=1}^{N(j)} D^j_n \theta^j_n \underline{\underline{\omega^j_n}}\end{aligned} \tag{9}$$

$$
\begin{aligned}
\underline{\underline{V^j}} = V^j(0) &- \sum_{k=1}^{|K(j)|} D_k^j t_k^j - \sum_{k=1}^{|K(j)|} D_k^j \theta_k^j \underline{\underline{\omega}}_k^j \\
&- \sum_{i=1}^{|I|} \frac{1}{2-\beta_i} g_i^j D_i [\grave{t}_i + \Delta t_i] \\
&- \sum_{i=1}^{|I|} \frac{1}{2-\beta_i} g_i^j D_i \theta_i' \underline{\underline{\omega}}_i - \sum_{i=1}^{|I|} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i [\grave{t}_i + \Delta t_i] \\
&- \sum_{i=1}^{|I|} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \hat{\theta}_i \underline{\underline{\omega}}_i + \sum_{i=1}^{|I_1|} f_i^j D_i \left[\grave{t}_i - (1 - \grave{x}_l + \grave{\theta}_i) \underline{\underline{\omega}}_i\right] \\
&- \sum_{m=1}^{|M(j)|} (1 - x_m^j + \theta_m^j) D_m^j \underline{\underline{\omega}}_m^j + \sum_{m=1}^{M(j)} D_m^j t_m^j \\
&- \sum_{n=1}^{|N(j)|} (1 - x_n^j + \theta_n^j) D_n^j \underline{\underline{\omega}}_n^j + \sum_{n=1}^{|N(j)|} D_n^j t_n^j
\end{aligned} \tag{10}
$$

Note that t_i' is converted into $\grave{t}_i + \Delta t_i$ from Eq. 5. The average inventory level of a storage unit is easily computed from Eq. 4.

$$
\begin{aligned}
\overline{V^j} = V^j(0) &+ \sum_{k=1}^{|K(j)|} \overline{\mathrm{UPSW}}(t; D_k^j, \omega_k^j, t_k^j, x_k^j,) \\
&+ \sum_{i=1}^{|I|} \overline{\mathrm{UPSW}}\left(t; \frac{1}{2-\beta_i} g_i^j D_i, \omega_i, t_i', x_i'\right) \\
&+ \sum_{i=1}^{|I|} \overline{\mathrm{UPSW}}\left(t; \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i, \omega_i, t_i', x_i'\right) \\
&- \sum_{i=1}^{|I|} \overline{\mathrm{UPSW}}(t; f_i^j D_i, \omega_i, \grave{t}_i, \grave{x}_l) \\
&- \sum_{m=1}^{|M(j)|} \overline{\mathrm{UPSW}}(t; D_m^j, \omega_m^j, t_m^j, x_m^j) \\
&- \sum_{n=1}^{|N(j)|} \overline{\mathrm{UPSW}}(t; D_n^j, \omega_n^j, t_n^j, x_n^j)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\overline{V^j} = V^j(0) &+ \sum_{k=1}^{|K(j)|} \frac{(1-x_k^j)}{2} D_k^j \underline{\underline{\omega}}_k^j - \sum_{k=1}^{|K(j)|} D_k^j t_k^j \\
&+ \sum_{i=1}^{|I|} \frac{(1-x_i')}{2} \frac{1}{2-\beta_i} g_i^j D_i \underline{\underline{\omega}}_i - \sum_{i=1}^{|I|} \frac{1}{2-\beta_i} g_i^j D_i [\grave{t}_i + \Delta t_i] \\
&+ \sum_{i=1}^{|I|} \frac{(1-x_i')}{2} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \underline{\underline{\omega}}_i - \sum_{i=1}^{|I|} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i [\grave{t}_i + \Delta t_i] \\
&- \sum_{i=1}^{|I|} \frac{(1-\grave{x}_l)}{2} f_i^j D_i \underline{\underline{\omega}}_i + \sum_{i=1}^{|I|} f_i^j D_i \grave{t}_i \\
&- \sum_{m=1}^{|M(j)|} \frac{(1-x_m^j)}{2} D_m^j \underline{\underline{\omega}}_m^j + \sum_{m=1}^{|M(j)|} D_m^j t_m^j \\
&- \sum_{n=1}^{|N(j)|} \frac{(1-x_n^j)}{2} D_n^j \underline{\underline{\omega}}_n^j + \sum_{n=1}^{|N(j)|} D_n^j t_n^j
\end{aligned} \tag{12}
$$

The objective function for the design of the batch-storage network is to minimize the annualized expectation of total cost, which consists of the setup cost of processes, the inventory holding cost of storage units, and the capital cost of the processes and storage units for a given time availability, size availability and occurrence number in a long cycle time of each process. Thus, the objective function can be expressed as:

$$
\begin{aligned}
\mathrm{TC} = &\sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \left[\frac{\eta_k^j A_k^j}{\tilde{\omega}_k^j} + a_k^j \overline{\overline{B}}_k^j\right] + \sum_{i=1}^{|I|} \left[\frac{\eta_i A_i}{\tilde{\omega}_i} + a_i \overline{\overline{B}}_i\right] \\
&+ \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} \left[\frac{\eta_n^j A_n^j}{\tilde{\omega}_n^j} + a_n^j \overline{\overline{B}}_n^j\right] + \sum_{j=1}^{|J|} \left[H^j \overline{V^j} + b^j \overline{\overline{V^j}}\right]
\end{aligned} \tag{13}
$$

Modifying Eq. 13 using $\frac{\eta A}{\tilde{\omega}} = \frac{\alpha A}{\underline{\underline{\omega}}}$ and $\overline{\overline{B}} = \frac{(2-\beta) D \underline{\underline{\omega}}}{\alpha}$ yields

$$
\begin{aligned}
\mathrm{TC} = &\sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \left[\frac{\alpha_k^j A_k^j}{\underline{\underline{\omega}}_k^j} + a_k^j \left(\frac{(2-\beta_k^j) D_k^j \underline{\underline{\omega}}_k^j}{\alpha_k^j}\right)\right] \\
&+ \sum_{i=1}^{|I|} \left[\frac{\alpha_i A_i}{\underline{\underline{\omega}}_i} + a_i \left(\frac{(2-\beta_i) D_i \underline{\underline{\omega}}_i}{\alpha_i}\right)\right] \\
&+ \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} \left[\frac{\alpha_n^j A_n^j}{\underline{\underline{\omega}}_n^j} + a_n^j \left(\frac{(2-\beta_n^j) D_n^j \underline{\underline{\omega}}_n^j}{\alpha_n^j}\right)\right] \\
&+ \sum_{j=1}^{|J|} \left[H^j \overline{V^j} + b^j \overline{\overline{V^j}}\right]
\end{aligned} \tag{14}
$$

Note that $\overline{V^j}$ and $\overline{\overline{V^j}}$ are further developed from Eqs. 9 and 12. The optimization constraints are no depletion of all storage units, $0 \le \underline{\underline{V^j}}$, where $\underline{\underline{V^j}}$ is obtained from Eq. 10. Design variables are $\underline{\underline{\omega}}_k^j, \underline{\underline{\omega}}_i, \underline{\underline{\omega}}_n^j, t_k^j, \grave{t}_i$, and t_n^j. Note that D_k^j, D_i, and D_n^j are considered constants in this first-level optimization problem. Table 1 summarizes the first-level optimization problem. Note that the constraint $0 \le \underline{\underline{V^j}}$ is a kind of worst-case scenario method. Consequently, it will generate pessimistic solutions.

Kuhn-Tucker Solution of the First-Level Optimization Problem

The procedure for solving the Kuhn-Tucker conditions of the first-level optimization problem is given in Appendix A. Optimal cycle times are as follows:

$$
\underline{\underline{\omega}}_k^j = \sqrt{\frac{\alpha_k^j A_k^j}{D_k^j \Psi_k^j}} \tag{15}
$$

$$
\underline{\underline{\omega}}_n^j = \sqrt{\frac{\alpha_n^j A_n^j}{D_n^j \Psi_n^j}} \tag{16}
$$

$$
\underline{\underline{\omega}}_i = \sqrt{\frac{\alpha_i A_i}{D_i \Psi_i}} \tag{17}
$$

where

$$
\Psi_k^j = [0.5 H^j + b^j](1 - x_k^j) + \frac{(2-\beta_k^j) a_k^j}{\alpha_k^j} + (H^j + 2b^j)\theta_k^j \tag{18}
$$

Table 1. First-Level Optimization Problem

Objective function	$\overline{TC}=\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\left[\frac{\alpha_k^j A_k^j}{\underline{\underline{\omega}}_k^j}+a_k^j\left(\frac{(2-\beta_k^j)D_k^j\underline{\underline{\omega}}_k^j}{\alpha_k^j}\right)\right]+\sum_{i=1}^{\lvert I\rvert}\left[\frac{\alpha_i A_i}{\underline{\underline{\omega}}_i}+a_i\left(\frac{(2-\beta_i)D_i\underline{\underline{\omega}}_i}{\alpha_i}\right)\right]+\sum_{j=1}^{\lvert J\rvert}\sum_{n=1}^{\lvert N(j)\rvert}\left[\frac{\alpha_n^j A_n^j}{\underline{\underline{\omega}}_n^j}+a_n^j\left(\frac{(2-\beta_n^j)D_n^j\underline{\underline{\omega}}_n^j}{\alpha_n^j}\right)\right]+\sum_{j=1}^{\lvert J\rvert}\left[H^j\overline{V^j}+b^j\overline{\overline{V^j}}\right]$
Design variables	$\underline{\underline{\omega}}_k^j,\underline{\underline{\omega}}_i,\underline{\underline{\omega}}_n^j,t_k^j,\grave{t}_i,$ and t_n^j
Constraints	$\underline{\underline{V^j}}\geq 0$
$\overline{\overline{V^j}}$, $\underline{\underline{V^j}}$, and $\bar{\bar{V}}^j$	$\overline{\overline{V^j}}=V^j(0)+\sum_{k=1}^{\lvert K(j)\rvert}(1-x_k^j+\theta_k^j)D_k^j\underline{\underline{\omega}}_k^j-\sum_{k=1}^{\lvert K(j)\rvert}D_k^jt_k^j+\sum_{i=1}^{\lvert I\rvert}(1-x_i'+\hat{\theta}_i)\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i[\grave{t}_i+\Delta t_i]+\sum_{i=1}^{\lvert I\rvert}(1-x_i'+\theta_i')\frac{1}{2-\beta_i}g_i^jD_i\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i[\grave{t}_i+\Delta t_i]+\sum_{i=1}^{\lvert I\rvert}f_i^jD_i\grave{t}_i+\sum_{i=1}^{\lvert I\rvert}f_i^jD_i\grave{\theta}_i\underline{\underline{\omega}}_i+\sum_{m=1}^{\lvert M(j)\rvert}D_m^jt_m^j+\sum_{m=1}^{\lvert M(j)\rvert}D_m^j\theta_m^j\underline{\underline{\omega}}_m^j+\sum_{n=1}^{\lvert N(j)\rvert}D_n^jt_n^j+\sum_{n=1}^{\lvert N(j)\rvert}D_n^j\theta_n^j\underline{\underline{\omega}}_n^j$
	$\underline{\underline{V^j}}=V^j(0)-\sum_{k=1}^{\lvert K(j)\rvert}D_k^jt_k^j-\sum_{k=1}^{\lvert K(j)\rvert}D_k^j\theta_k^j\underline{\underline{\omega}}_k^j-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i[\grave{t}_i+\Delta t_i]-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i\theta_i'\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i['t_i+\Delta t_i]-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i\hat{\theta}_i\underline{\underline{\omega}}_i+\sum_{i=1}^{\lvert I_1\rvert}f_i^jD_i\left[\grave{t}_i-(1-\grave{x}_i+\grave{\theta}_i)\underline{\underline{\omega}}_i\right]-\sum_{m=1}^{\lvert M(j)\rvert}(1-x_m^j+\theta_m^j)D_m^j\underline{\underline{\omega}}_m^j+\sum_{m=1}^{\lvert M(j)\rvert}D_m^jt_m^j-\sum_{n=1}^{\lvert N(j)\rvert}(1-x_n^j+\theta_n^j)D_n^j\underline{\underline{\omega}}_n^j+\sum_{n=1}^{\lvert N(j)\rvert}D_n^jt_n^j$
	$\overline{V^j}=V^j(0)+\sum_{k=1}^{\lvert K(j)\rvert}\frac{(1-x_k^j)}{2}D_k^j\underline{\underline{\omega}}_k^j-\sum_{k=1}^{\lvert K(j)\rvert}D_k^jt_k^j+\sum_{i=1}^{\lvert I\rvert}\frac{(1-x_i')}{2}\frac{1}{2-\beta_i}g_i^jD_i\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i[\grave{t}_i+\Delta t_i]+\sum_{i=1}^{\lvert I\rvert}\frac{(1-x_i')}{2}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i[\grave{t}_i+\Delta t_i]-\sum_{i=1}^{\lvert I\rvert}\frac{(1-\grave{x}_i)}{2}f_i^jD_i\underline{\underline{\omega}}_i+\sum_{i=1}^{\lvert I\rvert}f_i^jD_i\grave{t}_i-\sum_{m=1}^{\lvert M(j)\rvert}\frac{(1-x_m^j)}{2}D_m^j\underline{\underline{\omega}}_m^j+\sum_{m=1}^{\lvert M(j)\rvert}D_m^jt_m^j-\sum_{n=1}^{\lvert N(j)\rvert}\frac{(1-x_n^j)}{2}D_n^j\underline{\underline{\omega}}_n^j+\sum_{n=1}^{\lvert N(j)\rvert}D_n^jt_n^j$

$$\Psi_n^j=[0.5H^j+b^j](1-x_n^j)+\frac{(2-\beta_n^j)a_n^j}{\alpha_n^j}+(H^j+2b^j)\theta_n^j \tag{19}$$

$$\begin{aligned}\Psi_i=&\frac{(2-\beta_i)a_i}{\alpha_i}+(1-\grave{x}_l)\sum_{j=1}^{J}\left(\frac{H^j}{2}+b^j\right)f_i^j\\&+(1-x_i')\sum_{j=1}^{J}\left(\frac{H^j}{2}+b^j\right)\frac{1}{2-\beta_i}g_i^j\\&+(1-x_i')\sum_{j=1}^{J}\left(\frac{H^j}{2}+b^j\right)\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j\\&+\sum_{j=1}^{J}(H^j+2b^j)\left(f_i^j\grave{\theta}_i+\frac{1}{2-\beta_i}g_i^j\theta_i'+\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j\hat{\theta}_i\right)\end{aligned} \tag{20}$$

The optimal objective value is:

$$\begin{aligned}{}^*\mathrm{TC}(D_k^j,D_n^j,D_i)=&2\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\sqrt{\alpha_k^jA_k^j\Psi_k^jD_k^j}+2\sum_{i=1}^{\lvert I\rvert}\sqrt{\alpha_iA_i\Psi_iD_i}\\&+2\sum_{j=1}^{\lvert J\rvert}\sum_{n=1}^{\lvert N(j)\rvert}\sqrt{\alpha_n^jA_n^j\Psi_n^jD_n^j}+\sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}\left(\frac{H^j}{2}+b^j\right)\\&\times(1-x_m^j)\underline{\underline{\omega}}_m^jD_m^j+\sum_{r=1}^{\lvert R\rvert}\sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}\theta_m^jD_m^j\underline{\underline{\omega}}_m^j(H^j+2b^j)\end{aligned} \tag{21}$$

The contribution of process i to the objective values is:

$$\text{Cost of Process } i=2\sqrt{\alpha_iA_i\Psi_iD_i}. \tag{22}$$

Optimal startup times are derived from the equality of the constraint $0\leq\underline{\underline{V^j}}$, where $\underline{\underline{V^j}}$ is given by Eq. 10.

$$\begin{aligned}&\sum_{k=1}^{\lvert K(j)\rvert}D_k^jt_k^j+\sum_{i=1}^{I}\left(\frac{1}{2-\beta_i}g_i^j+\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j-f_i^j\right)D_i\grave{t}_i-\sum_{n=1}^{N(j)}D_n^jt_n^j\\&=V^j(0)-\sum_{m=1}^{\lvert M(j)\rvert}(1-x_m^k+\theta_m^j)D_m^j\underline{\underline{\omega}}_m^j+\sum_{m=1}^{\lvert M(j)\rvert}D_m^jt_m^j\\&\quad-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i\Delta t_i-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i\Delta t_i\\&\quad-\sum_{k=1}^{\lvert K(j)\rvert}D_k^j\theta_k^j\underline{\underline{\omega}}_k^j-\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i}g_i^jD_i\theta_i'\underline{\underline{\omega}}_i-\sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^jD_i\theta_i'\underline{\underline{\omega}}_i\\&\quad-\sum_{i=1}^{\lvert I\rvert}f_i^jD_i(1-\grave{x}_i+\grave{\theta}_i)\underline{\underline{\omega}}_i-\sum_{n=1}^{\lvert N(j)\rvert}(1-x_n^j+\theta_n^j)D_n^j\underline{\underline{\omega}}_n^j\end{aligned} \tag{23}$$

The optimal storage sizes, derived from Eqs. 23 and 9, are:

Table 2. Second-Level Optimization Problem

Objective function	${}^{*}\mathrm{TC}(D_k^j, D_n^j, D_i) = 2\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\sqrt{\alpha_k^j A_k^j \Psi_k^j D_k^j} + 2\sum_{i=1}^{\lvert I\rvert}\sqrt{\alpha_i A_i \Psi_i D_i} + 2\sum_{j=1}^{\lvert J\rvert}\sum_{n=1}^{\lvert N(j)\rvert}\sqrt{\alpha_n^j A_n^j \Psi_n^j D_n^j} + \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}\left(\frac{H^j}{2} + b^j\right)(1 - x_m^j)\underline{\underline{\omega}}_m^j D_m^j + \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}\theta_m^j D_m^j \underline{\underline{\omega}}_m^j (H^j + 2b^j)$
Design variables	D_n^j, and D_i
Constraints	$\sum_{i=1}^{\lvert I\rvert}\frac{1}{2-\beta_i} g_i^j D_i + \sum_{i=1}^{\lvert I\rvert}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j D_i + \sum_{k=1}^{\lvert K(j)\rvert} D_k^j = \sum_{i=1}^{\lvert I\rvert} f_i^j D_i + \sum_{m=1}^{\lvert M(j)\rvert} D_m^j + \sum_{n=1}^{\lvert N(j)\rvert} D_n^j$

$$\overline{\overline{V^j}} = \sum_{i=1}^{|I|}\left[(1 - \grave{x}_i)f_i^j + (1 - x_i')\frac{1}{2-\beta_i}g_i^j + (1 - x_i')\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j\right] \times D_i\underline{\underline{\omega}}_i + \sum_{k=1}^{|K(j)|}(1 - x_k^j)D_k^j\underline{\underline{\omega}}_k^j + \sum_{m=1}^{|M(j)|}(1 - x_m^j)D_m^j\underline{\underline{\omega}}_m^j + \sum_{n=1}^{|N(j)|}(1 - x_n^j)D_n^j\underline{\underline{\omega}}_n^j + 2\sum_{i=1}^{|I|}\frac{1}{2-\beta_i}g_i^j D_i\theta_i'\underline{\underline{\omega}}_i + 2\sum_{i=1}^{|I|}\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j D_i\hat{\theta}_i\underline{\underline{\omega}}_i + 2\sum_{i=1}^{|I|}f_i^j D_i\,\grave{\theta}_i\underline{\underline{\omega}}_i + 2\sum_{k=1}^{|K(j)|}D_k^j\theta_k^j\underline{\underline{\omega}}_k^j + 2\sum_{m=1}^{|M(j)|}\theta_m^j D_m^j\underline{\underline{\omega}}_m^j + 2\sum_{n=1}^{|N(j)|}\theta_n^j D_n^j\underline{\underline{\omega}}_n^j \tag{24}$$

The contribution of process i to storage j is:

$$\left[(1 - \grave{x}_i)f_i^j + (1 - x_i')\frac{1}{2-\beta_i}g_i^j + (1 - x_i')\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j\right]D_i\underline{\underline{\omega}}_i + \frac{2}{2-\beta_i}g_i^j D_i\theta_i'\underline{\underline{\omega}}_i + 2\frac{1-\beta_i}{2-\beta_i}\hat{g}_i^j D_i\hat{\theta}_i\underline{\underline{\omega}}_i + 2f_i^j D_i\grave{\theta}_i\underline{\underline{\omega}}_i \tag{25}$$

To compute optimal average flow rates D_k^j, D_n^j, and D_i, the second-level optimization problem with Eq. 21 as the objective function and Eq. 6 as the constraint, summarized in Table 2, should be solved numerically. Although the optimization problem is separated into a two-level parametric optimization problem, the Kuhn-Tucker conditions of the original problem and the two-level problem are the same.[4] In other words, the Kuhn-Tucker conditions of the first-level optimization problem produce an explicit analytical solution and the original optimization problem can be reduced to the second-level optimization problem by eliminating the design variables of the first-level optimization problem. The two-level parametric approach yields a global optimum in as much as the second-level optimization problem converges to its global optimum point. The nonlinear objective function in Eq. 21 is a separable concave function (square root) and can be linearized in a piecewise manner by using the specially ordered sets (SOS) formulation. Appendix B summarizes the computational procedure. Simulation results showed that the computation time using SOS formulation takes about seven times of the computation time of the relaxed linear programming model.[7] The second-level optimization problem can be replaced with ordinary planning model with linear programming without damaging the optimality of the first-level problem.

Design Examples

Let us consider a process that produces two products and one waste material from three feedstock materials in the batch-storage network. The configuration of this process is shown in Figure 7, which also includes most of the input data required for the computations. D_1,a_1, and A_1 are the average flow rate through process 1, the annualized capital cost per capacity of process 1, and the setup cost of process 1, respectively. b^j and H^j $j = 1,2, \ldots, 6$ are the annualized capital cost per capacity of storage unit j and the inventory holding cost of storage unit j, respectively. Feedstock composition f_1^1,f_1^2, and f_1^3 are 0.5, 0.3, and 0.2, respectively, and product yield g_1^5 and g_1^6 are 0.1 and 0.9, respectively. Waste material yield $\hat{g}_1^4 = 1$. Provided that the time availability α_1, size availability β_1 and occurrence number η_1 are given, the optimal cycle time, optimal batch size, optimal cost of the process, and optimal storage size can be computed. The computation starts by computing the theta values using Eq. 3:

$$\grave{\theta}_1 = \left(\frac{1}{\alpha_1} - 1\right)\eta_1, \quad \theta_1' = \left(\frac{1}{\alpha_1} - 1\right)\eta_1 + \frac{(\alpha_1 - \beta_1)^+}{\alpha_1}(0.5\eta_1),$$
$$\hat{\theta}_1 = \left(\frac{1}{\alpha_1} - 1\right)\eta_1 + 0.5\eta_1$$

Next, the aggregated cost Ψ_1 is computed using Eq. 20. Note that $\grave{x}_1$ and $x_1' = 0$.

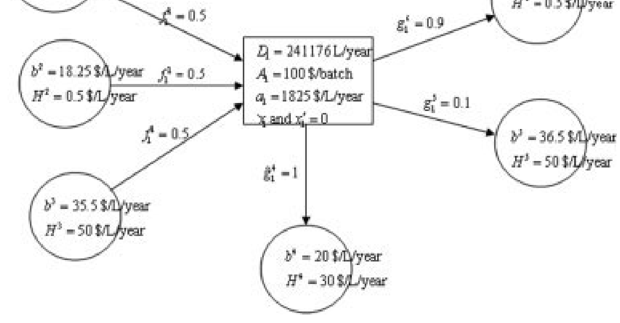

Figure 7. Design example.

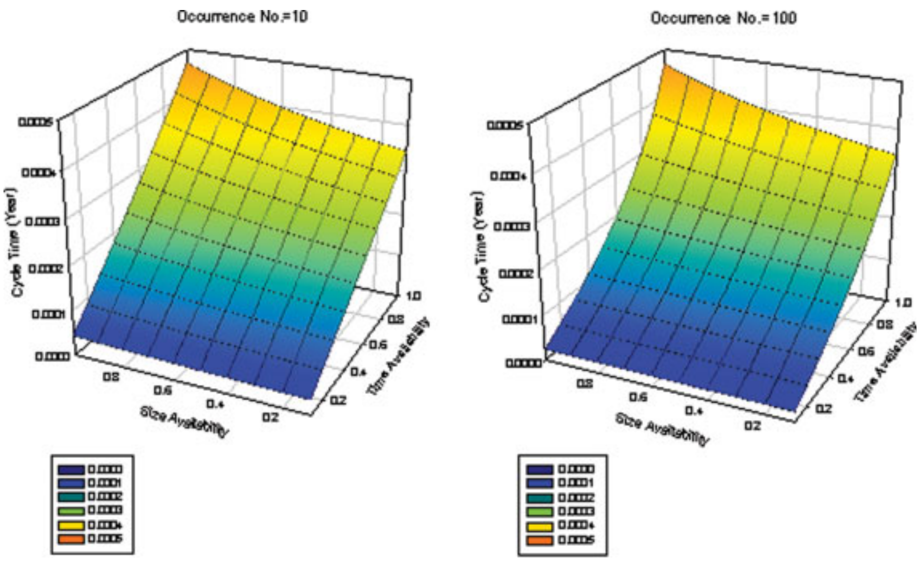

Figure 8. Sensitivity analysis with two fixed occurrence numbers—cycle time.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$$
\begin{aligned}
\Psi_1 = {} & \frac{(2-\beta_1)a_1}{\alpha_1} + \left(\frac{H^1}{2}+b^1\right)f_1^1 + \left(\frac{H^2}{2}+b^2\right)f_1^2 \\
& + \left(\frac{H^3}{2}+b^3\right)f_1^3 + \left(\frac{H^4}{2}+b^4\right)\frac{1-\beta_1}{2-\beta_1}\hat{g}_1^4 \\
& + \left(\frac{H^5}{2}+b^5\right)\frac{g_1^5}{2-\beta_1} + \left(\frac{H^6}{2}+b^6\right)\frac{g_1^6}{2-\beta_1} \\
& + \left(H^1+2b^1\right)f_1^1\grave{\theta}_1 + \left(H^2+2b^2\right)f_1^2\grave{\theta}_1 \\
& + \left(H^3+2b^3\right)f_1^3\grave{\theta}_1 + \left(H^4+2b^4\right)\frac{1-\beta_1}{2-\beta_1}\hat{g}_1^4\hat{\theta}_1 \\
& + \left(H^5+2b^5\right)\frac{g_1^5\theta_1'}{2-\beta_1} + \left(H^6+2b^6\right)\frac{g_1^6\theta_1'}{2-\beta_1}
\end{aligned}
$$

Then, the optimal cycle time ω_1 from Eq. 17, optimal batch size $\overline{B}_1 = \frac{D_1\omega_1}{\alpha_1}$, optimal cost of process i from Eq. 22, and optimal storage size from Eq. 25 can be computed. The computation results are given in Figures 8–11 with respect to various values of α_1,β_1, and η_1.

Figure 8 shows the dependencies of the optimal solutions for cycle time on time availability and size availability for two fixed values of occurrence number, 10 and 100. The optimal cycle time increases as both availabilities increase, with the time availability effect being stronger than the size availability effect. The effect of occurrence number is weaker than those of the two availabilities.

Figure 9 shows the dependencies of the optimal solutions for batch size on time availability and size availability for

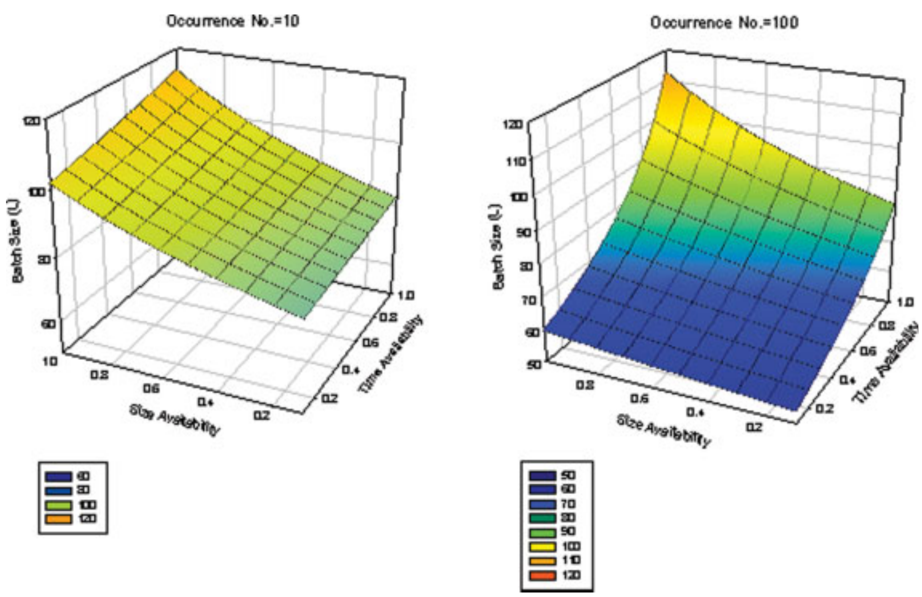

Figure 9. Sensitivity analysis with two fixed occurrence numbers—batch size.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

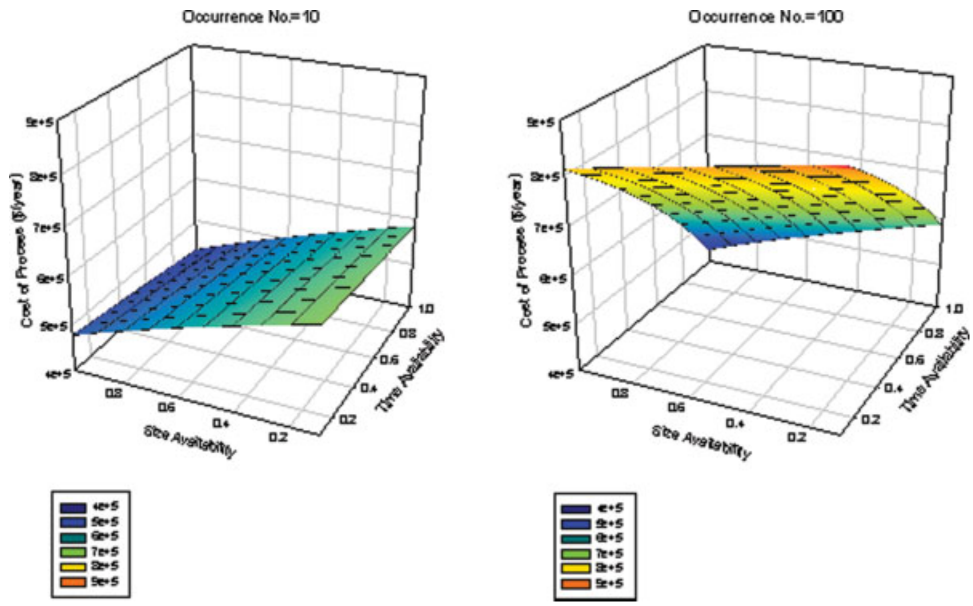

Figure 10. Sensitivity analysis with two fixed occurrence numbers—cost of process.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

two fixed values of the occurrence number, 10 and 100. The optimal batch size increases as both availabilities increase. The relative strengths of the effects of the two availabilities are indecisive. As the occurrence number increases, the value of the optimal batch size becomes more widely spread.

Figure 10 shows the dependencies of the optimal solutions for the cost of the process on time availability and size availability for two fixed values of occurrence number, 10 and 100. The optimal cost of the process decreases as both availabilities increase, and increases as the occurrence number increases.

Figure 11 shows the dependencies of the optimal solutions for storage size on time availability and size availability for two fixed values of the occurrence number, 10 and 100. The variation in the optimal storage size as a function of the time availability and size availability is highly nonlinear. The optimal storage size shows a maximum as the time availability approaches 0 and the size availability approaches 1. Optimal storage size shows minimum as both availabilities approaches 1. The optimal storage size increases as the occurrence number increases.

Concluding Remarks

This study deals with determining the optimal sizes of batch processes and storage units interconnected in a general

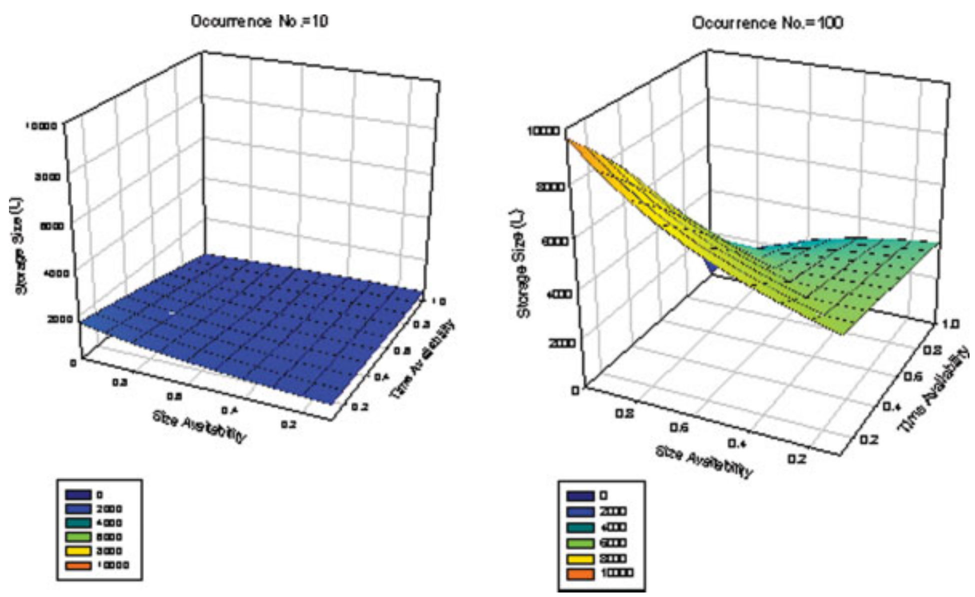

Figure 11. Sensitivity analysis with two fixed occurrence numbers—storage size.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

network structure when the processes are subject to joint uncertainties of operating time and batch quantity. Waste regeneration and disposal processes were included into the network to treat materials from failed batches. An unique graphical method was used to find the upper and lower bounds and average of material flows susceptible to short-term joint random variations in the cycle time and batch size. In the definition of the random properties, availabilities and occurrence number were introduced as input parameters instead of more widely used parameters such as the mean and variance. The availability is commonly used in process reliability analysis methods such as failure modes and effects analysis, and the occurrence number is proportional to the variance. These parameters were chosen as they are more practical and easier to estimate based on human perception. The optimization problem consisted of minimizing the expected sum of the setup cost, capital cost of processes storage units, and inventory holding cost under the constraints of meeting random product demand and no depletion of storage materials. The stochastic version of the PSW model with unique graphical analysis provided analytical solutions of the optimization problem. These analytical solutions greatly reduce the computational burden, which is the major achievement of this study. The analytical optimal solutions made it possible to conduct sensitivity analysis with respect to the input parameters, time availability and size availability with two fixed values of occurrence number.

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Notation

a_k^j = annualized capital cost of raw material purchasing facility, $/L/year
a_i = annualized capital cost of unit i, $/L/year
a_n^j = annualized capital cost of waste disposal facility, $/L/year
A_k^j = ordering cost of feedstock materials, $/order
A_i = ordering cost of noncontinuous units, $/batch
A_n^j = disposal cost of waste materials, $/batch
b^j = annualized capital cost of storage facility, $/L/year
$\mathbf{B}_{(l)}$ = random variable of batch size, L/batch
$\underline{\overline{B}}$ = general description of batch size of any process, L/batch
$\overline{B}$ = general description of average batch size of any process, L/batch
$\underline{\underline{B}}$ = general description of minimum batch size of any process, L/batch
$\overline{\overline{B}}$ = general description of maximum batch size of any process, L/batch
$\overline{\overline{B}}_i$ = maximum batch size of process i, L/batch
$\overline{B}_i$ = mean of batch size of process i, L/batch
$\underline{\underline{B}}_i$ = minimum batch size of process i, L/batch
$\mathbf{B}'_{\mathbf{i(l)}}$ = random l-th batch size of product flow of process i, L/batch
B_k^j = raw material order size, L/batch
$\overline{B}_k^j$ = mean of raw material order size, L/batch
$\overline{\overline{B}}_k^j$ = maximum batch size of raw material purchase, L/batch
B_m^j = finished product demand batch size, L/batch
$\overline{B}_m^j$ = mean of batch size of finished product demand, L/batch
B_n^j = batch size of waste disposal sink, L/batch
$\overline{B}_n^j$ = mean of batch size of waste disposal sink, L/batch
$\overline{\overline{B}}_n^j$ = maximum batch size of waste disposal sink, L/batch
d = general description of total dead time within a long cycle time, year
D = general description of average flow rates of any process, L/year
D_i = average material flow rate through process i, L/year
D_k^j = average material flow rate of raw material supply, L/year
D_m^j = average material flow rate of customer demand, L/year
D_n^j = average material flow rate of waste disposal, L/year
f_i^j = feedstock composition of process i
$\mathbf{F}(t)$ = random batch flow, L
g_i^j = product yield of process i
$\hat{g}_i^j$ = waste material yield of process i
H^j = annual inventory holding costs, $/L/year
I = batch process set
J = storage set
$K(j)$ = raw material supplier set for material j
$M(j)$ = consumer set for material j
$N(j)$ = waste disposal sink set for material j
t' = general description of startup time, year
t_m^j = startup time of customer demand, year
t_n^j = startup time of waste disposal, year
$\grave{t}_i$ = startup time of feedstock feeding to batch process i, year
t_i' = startup time of product discharging from batch process i, year
t_k^j = startup time of raw material purchasing, year
Δt_i = time delay between feed flow and product flow of process i, year
TC = total cost, $/year
$\underline{\underline{\mathrm{UPSW}}}$ = defined by Eq. 1
$\overline{\overline{\mathrm{UPSW}}}$ = defined by Eq. 2
$\overline{\mathrm{UPSW}}$ = defined by Eq. 4
$\overline{\overline{V^j}}$ = upper bound of inventory hold-up, L
$\underline{\underline{V^j}}$ = lower bound of inventory hold-up, L
$V^j(t)$ = inventory hold-up, L
$V^j(\underline{0})$ = initial inventory hold-up, L
$\overline{V^j}$ = time averaged inventory hold-up, L
x = general description of storage operation time fraction
x_k^j = storage operation time fraction of purchasing raw materials
$\grave{x}_i$ = storage operation time fraction of feeding to process unit i
x_i' = storage operation time fraction of discharging from process unit i
x_m^j = storage operation time fraction of finished product demand
x_n^j = storage operation time fraction of waste disposal flow

Greek letters

α = general description of time availability
α_k^j = time availability of raw material purchase flow
α_i = time availability of process i
α_m^j = time availability of finished product sales flow
α_n^j = time availability of waste disposal flow
β = general description of size availability
β_i = size availability of process i
β_k^j = size availability of raw material purchase
β_n^j = size availability of waste disposal sink
δ_1,δ_2 = confidence limits
$\varepsilon_1,\varepsilon_2$ = convergence limit
λ^j = Largrangian multiplier of Kuhn-Tucker conditions
η = general description of occurrence number
η_i = occurrence number of process i
η_k^j = occurrence number of raw material purchase
η_m^j = occurrence number of finished product demand
η_n^j = occurrence number of waste disposal sink
θ = defined as Eq. 3
θ_k^j = defined by Eq. 3
θ_m^j = defined by Eq. 3
θ_n^j = defined by Eq. 3
θ_i' = defined by Eq. 3
$\grave{\theta}_i = (\frac{1}{\alpha_i} = 1)\eta_i$
$\hat{\theta}_i = (\frac{1}{\alpha_i} = 1)\eta_i + (0.5\eta_i)$
ω = general description of cycle time of any process, year
$\boldsymbol{\omega}_{(\mathbf{l})}$ = (l)-th random variable of cycle time of any process, year
$\boldsymbol{\omega}_{\mathbf{i(l)}}$ = (l)-th random variable of cycle time of process i, year

$\underline{\underline{\omega}}$ = general description of minimum cycle time of any process, year
ω = general description of average cycle time of any process, year
$\widetilde{\omega}$ = general description of long cycle time, year
ω^j_m = cycle time of customer demand, year
ω^j_n = cycle time of waste disposal flow, year
ω^j_k = cycle time of raw material purchasing, year
ω_i = cycle time of process i, year
$\underline{\underline{\omega}}^j_k$ = lower bound of cycle time of raw material purchase, year
$\underline{\underline{\omega}}^j_m$ = lower bound of cycle time of finished product demand flow, year
$\underline{\underline{\omega}}^j_n$ = lower bound of cycle time of waste disposal flow, year
$\widetilde{\omega}_i$ = long cycle time of process i, year
$\widetilde{\omega}^j_k$ = long cycle time of raw material purchasing, year
$\widetilde{\omega}^j_m$ = long cycle time of customer demand, year
$\widetilde{\omega}^j_n$ = long cycle time of waste disposal flow, year
Ψ^j_k = aggregated cost defined by Eq. 18
Ψ^j_n = aggregated cost defined by Eq. 19
Ψ_i = aggregated cost defined by Eq. 20

Subscripts

i = batch process index
k = index of raw material vendors
(l) = index of batch sequence of occurrence
m = index of finished product customers
n = index of waste disposal sinks

Superscript

j = storage index

Special functions

int[.] = truncation function to make integer
res[.] = positive residual function to be truncated
Var(.) = variance
$|X|$ = number of elements in set X
$P\{.\}$ = probability
$\underline{X}$ = average of X
$\overline{\overline{X}}$ = upper bound of X
$\underline{\underline{X}}$ = lower bound of X
$(X)^{+}$ = max$\{0,x\}$

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Appendix A: Kuhn-Tucker Solution to the First-Level Optimization Problem

The partial derivatives of $\overline{\overline{V^j}}$, $\underline{\underline{V^j}}$, and $\overline{V}^j$ with respect to design variables $\underline{\underline{\omega}}^j_k, \underline{\underline{\omega}}_i, \underline{\underline{\omega}}^j_n, t^j_k, \grave{t}_i$, and t^j_n are as follows:

$$\frac{\partial \overline{\overline{V^j}}}{\partial t^j_k} = -D^j_k, \quad \frac{\partial \overline{\overline{V^j}}}{\partial \grave{t}_i} = \left(f^j_i - \frac{1}{2-\beta_i} g^j_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i \right) D_i,$$

$$\frac{\partial \overline{\overline{V^j}}}{\partial t^j_n} = D^j_n, \quad \frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}^j_k} = (1 - x^j_k + \theta^j_k) D^j_k, \quad \frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}^j_n} = \theta^j_n D^j_n$$

$$\frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}_i} = (1 - x'_i) \frac{1}{2-\beta_i} g^j_i D_i - \frac{1}{2-\beta_i} g^j_i D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} + f^j_i \grave{\theta}_i D_i + \frac{1}{2-\beta_i} g^j_i \theta'_i D_i + (1 - x'_i) \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} + \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i \hat{\theta}_i D_i \tag{A1}$$

$$\frac{\partial \underline{\underline{V^j}}}{\partial t^j_k} = -D^j_k, \quad \frac{\partial \underline{\underline{V^j}}}{\partial' t_i} = \left(f^j_i - \frac{1}{2-\beta_i} g^j_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i \right) D_i,$$

$$\frac{\partial \underline{\underline{V^j}}}{\partial t^j_n} = D^j_n, \quad \frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}^j_k} - \theta^j_k D^j_k, \quad \frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}^j_n} = -(1 - x^j_n + \theta^j_n) D^j_n$$

$$\frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}_i} = -(1 - \grave{x}_i) f^j_i D_i - \frac{1}{2-\beta_i} g^j_i D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} - f^j_i \grave{\theta}_i D_i - \frac{1}{2-\beta_i} g^j_i \theta'_i D_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} - \frac{1-\beta_i}{2-\beta_i} \hat{g}^j_i \hat{\theta}_i D_i \tag{A2}$$

$$\frac{\partial \overline{V^j}}{\partial t_k^j} = -D_k^j, \quad \frac{\partial \overline{V^j}}{\partial \grave{t}_i} = \left(f_i^j - \frac{1}{2-\beta_i} g_i^j - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j \right) D_i,$$

$$\frac{\partial \overline{V^j}}{\partial t_n^j} = D_n^j, \quad \frac{\partial \overline{V^j}}{\partial \underline{\underline{\omega}}_k^j} = \frac{(1-x_k^j)}{2} D_k^j, \quad \frac{\partial \overline{V^j}}{\partial \underline{\underline{\omega}}_n^j} = -\frac{(1-x_n^j)}{2} D_n^j$$

$$\frac{\partial \overline{V^j}}{\partial \underline{\underline{\omega}}_i} = -\frac{(1-\grave{x}_i)}{2} f_i^j D_i + \frac{(1-x_i')}{2} \frac{1}{2-\beta_i} g_i^j D_i$$

$$- \frac{1}{2-\beta_i} g_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} + \frac{(1-x_i')}{2} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \tag{A3}$$

The Lagrangean for the optimization problem to minimize Eq. 14 subject to $0 \le \underline{\underline{V^j}}$ with respect to $\underline{\underline{\omega}}_k^j$, $\underline{\underline{\omega}}_i$, $\underline{\underline{\omega}}_n^j$, $t_k^j, \grave{t}_i$, and t_n^j is;

$$L(\underline{\underline{\omega}}_k^j, \underline{\underline{\omega}}_i, \underline{\underline{\omega}}_n^j, t_k^j, \grave{t}_i, t_n^j) = TC - \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j \underline{\underline{V^j}} \tag{A4}$$

where $\underline{\underline{\lambda}}^j$ is the Lagrange multiplier. Kuhn-Tucker conditions give:

$$\frac{\partial L}{\partial t_k^j} = H^j \frac{\partial \overline{V^j}}{\partial t_k^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial t_k^j} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial t_k^j} = -(H^j + b^j) D_k^j + \underline{\underline{\lambda}}^j D_k^j = 0 \tag{A5}$$

$$\frac{\partial L}{\partial t_n^j} = H^j \frac{\partial \overline{V^j}}{\partial t_n^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial t_n^j} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial t_n^j} = (H^j + b^j) D_n^j - \underline{\underline{\lambda}}^j D_n^j = 0 \tag{A6}$$

$$\frac{\partial L}{\partial \grave{t}_i} = \sum_{j=1}^{|J|} \left[\left[H^j \frac{\partial \overline{V^j}}{\partial \grave{t}_i} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \grave{t}_i} \right] - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \grave{t}_i} \right]$$

$$= \sum_{j=1}^{|J|} (H^j + b^j) \left(f_i^j - \frac{1}{2-\beta_i} g_i^j - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j \right) D_i$$

$$- \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j \left(f_i^j - \frac{1}{2-\beta_i} g_i^j - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j \right) D_i = 0 \tag{A7}$$

$$\frac{\partial L}{\partial \underline{\underline{\omega}}_k^j} = -\frac{\alpha_k^j A_k^j}{(\underline{\underline{\omega}}_k^j)^2} + \frac{(2-\beta_k^j) a_k^j D_k^j}{\alpha_k^j} + H^j \frac{\partial \overline{V_j}}{\partial \underline{\underline{\omega}}_k^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}_k^j} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}_k^j}$$

$$= -\frac{\alpha_k^j A_k^j}{\left(\underline{\underline{\omega}}_k^j\right)^2} + \left[\left(0.5 H^j (1-x_k^j) + b^j (1 - x_k^j + \theta_k^j) \right) \right.$$

$$\left. + \frac{(2-\beta_k^j) a_k^j}{a_k^j} \right] D_k^j + \underline{\underline{\lambda}}^j \theta_k^j D_k^j = 0 \tag{A8}$$

$$\frac{\partial L}{\partial \underline{\underline{\omega}}_n^j} = -\frac{\alpha_n^j A_n^j}{(\underline{\underline{\omega}}_n^j)^2} + \frac{(2-\beta_n^j) a_n^j D_n^j}{\alpha_n^j} + H^j \frac{\partial \overline{V^j}}{\partial \underline{\underline{\omega}}_n^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}_n^j} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}_n^j}$$

$$= -\frac{\alpha_n^j A_n^j}{(\underline{\underline{\omega}}_n^j)^2} + \left[(-0.5 H^j (1-x_n^j) + b^j \theta_n^j) + \frac{(2-\beta_n^j) a_n^j}{\alpha_n^j} \right] D_n^j$$

$$+ \underline{\underline{\lambda}}^j (1 - x_n^j + \theta_n^j) D_n^j = 0 \tag{A9}$$

$$\frac{\partial L}{\partial \underline{\underline{\omega}}_i} = -\frac{\alpha_i A_i}{(\underline{\underline{\omega}}_i)^2} + \frac{(2-\beta_i) a_i D_i}{\alpha_i}$$

$$+ \sum_{j=1}^{|J|} \left[H^j \frac{\partial \overline{V^j}}{\partial \underline{\underline{\omega}}_i} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \underline{\underline{\omega}}_i} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \underline{\underline{\omega}}_i} \right]$$

$$= -\frac{\alpha_i A_i}{(\underline{\underline{\omega}}_i)^2} + \frac{(2-\beta_i) a_i D_i}{\alpha_i} + \sum_{j=1}^{|J|} H^j \left\{ -\frac{(1-\grave{x}_i)}{2} f_i^j D_i \right.$$

$$\left. + \frac{(1-x_i')}{2} \frac{1}{2-\beta_i} g_i^j D_i - \frac{1}{2-\beta_i} g_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \right\}$$

$$+ \sum_{j=1}^{|J|} H^j \left\{ \frac{(1-x_i')}{2} \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \right\}$$

$$+ \sum_{j=1}^{|J|} b^j \left\{ (1-x_i') \frac{1}{2-\beta_i} g_i^j D_i - \frac{1}{2-\beta_i} g_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \right.$$

$$\left. + f_i^j{}' \theta_i D_i + \frac{1}{2-\beta_i} g_i^j \theta_i' D_i \right\}$$

$$+ \sum_{j=1}^{|J|} b^j \left\{ (1-x_i') \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \right.$$

$$\left. + \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j \hat{\theta}_i D_i \right\}$$

$$- \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j \left\{ -(1-\grave{x}_i) f_i^j D_i - \frac{1}{2-\beta_i} g_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} \right.$$

$$\left. - f_i^j{}' \theta_i D_i - \frac{1}{2-\beta_i} g_i^j \theta_i' D_i \right\}$$

$$- \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j \left\{ -\frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j D_i \frac{\partial \Delta t_i}{\partial \underline{\underline{\omega}}_i} - \frac{1-\beta_i}{2-\beta_i} \hat{g}_i^j \hat{\theta}_i D_i \right\} = 0 \tag{A10}$$

$$\underline{\underline{\lambda}}^j \underline{\underline{V^j}} = 0 \tag{A11}$$

Solving Eqs. A5–A7 gives;

$$\underline{\underline{\lambda}}^j = H^j + b^j \tag{A12}$$

Solving Eqs. A8–A10 with Eq. A12 gives Eqs. 15–17 in the main text. Solving Eq. A11 gives Eq. 21 in the main text.

Appendix B: Piecewise Linear Approximation of Separable Function by Using Specially Ordered Sets

The following equations show how to linearize $\sqrt{\alpha_i A_i \Psi_i D_i}$ in a piecewise manner with respect to D_i as a suitable formulation for specially ordered sets.

$$\begin{aligned}
&\sum_n \lambda_i^n = 1\\
&D_i = \sum_n \lambda_i^n D_i^n\\
&\sqrt{\alpha_i A_i \Psi_i D_i} = \sum_n \lambda_i^n \sqrt{\alpha_i A_i \Psi_i D_i^n}\\
&D_i^n = D_i^{\min} + \frac{n}{N}(D_i^{\max} - D_i^{\min})\\
&(n = 0, 1, 2, \ldots, N)
\end{aligned} \tag{B1}$$

where D_i and λ_i^n are nonnegative variables and the other variables are treated as parameters. λ_i^n is specially ordered sets that have the property of $\lambda_i^{n\prime} + \lambda_i^{n'+1} = 1$ and λ_i^0, λ_i^1, …, $\lambda_i^{n'-1}$, $\lambda_i^{n'+2}$, …, $\lambda_i^N = 0$. GAMS/CPLEX has the special ability to deal with specially ordered sets by declaring SOS2 VARIABLES.[20] $N = 5$ is sufficient to approximate a square root function.

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